TOPIC :→ Uniform Convergence (Real Analysis)

Introduction :→

We consider infinite series or sequence whose terms are all real functions of x where x is variable over a non-empty subset of R. (R = Set of Real No).

Let $\{f_n\}$. $(n = 1, 2, 3, \ldots)$ be a sequence of a real functions defined on a set E where $E \subset R$.

Also suppose that the sequence $\{f_n\}$ of real numbers converges for every $x \in E$

i.e $\lim_{n\to\infty} f_n(x) = f(x) \quad x \in E$ — (I)

In view of the above definition we say that $\{f_n\}$ converges to f on E and f is called the limit of $\{f_n\}$

or

$\lim_{n\to\infty} f_n(x) = f(x)$ means that for every $\epsilon > 0$ there exists a +ve integer m such that

$|f(x) - f_n(x)| < \epsilon$ for all $n \geq m$ — (II)

In this case the choice of m depends on ϵ as well as on x.

So, we say that $\{f_n\}$ converges to f point-wise on E

Similarly, if $\sum^{\infty} f_n(x)$ converges for every $x \in E$ and if we define

$f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in E$ then f is called the sum of the series of function.

If it is possible to choose a single m so that the inequality

$$|f(x)-f_n(x)| < \varepsilon \quad \text{for all } n \geq m$$

is satisfied for all values of x in E

the possibility of such m means m is a function of ε and x both considered as a function of x alone possesses as upper ~~bound~~ bound $m(\varepsilon) = m$ this is however not possible.

When this property exists, we say that the sequence $\{f_n(x)\}$ converges to the limit $f(x)$ uniformly on the set E.

* <u>Uniformly Convergence :-></u>

The sequence $\{f_n(x)\}$ is said to converge uniformly to the limit $f(x)$ on the set E, if given $\varepsilon > 0$, we can find a +ve integer m independent of x such that for all values of $n > m$ and for all x on E

$$|f_n(x) - f(x)| < \varepsilon$$

In this case we say that f is the uniform limit of the sequence $\{f_n\}$

In the case of point-wise convergence, for $\varepsilon > 0$ and for each $x \in E$ there corresponds a +ve integer m whereas in the case of uniform convergence, for each $\varepsilon > 0$ we can find a +ve integer m (m depends only on ε) such that the inequality (2) is satisfied for all $x \in E$.

Remarks :→ Every uniformly convergent sequence is point-wise convergent but a point-wise convergent sequence is not necessarily uniform convergent.

Uniformly Convergent Series :→

The series $\Sigma u_n(x)$ converges uniformly on E if the sequence $\{f_n(x)\}$ of partial sum defined by $f_n(x) = u_1(x) + u_2(x) + \dots + u_n(x)$ converges uniformly on E.

Ex :→ Prove that the sequence of functions $\{f_n(x)\}$ of real valued function defined on $[0,1]$ by

$$f_n(x) = \frac{nx}{1+n^2x^2} \quad \text{for } x \in [0,1]$$

is point-wise convergent but not uniformly convergen.

Solution :→ for each fixed $x \in [0,1]$, we have

$$f_n(x) = \frac{nx}{1+n^2x^2} = \frac{nx}{n^2\left[\frac{1}{n^2}+x^2\right]}$$

$$= \frac{x/n}{\frac{1}{n^2}+x^2} \to 0 \quad \text{as } n \to \infty$$

This the sequence $f_n(x)$ is point-wise convergent to the function f defined by $f(x) = 0$ for all $x \in E$.

Now, if $x = 0$ then $f_n(x) = 0$

$\therefore f(x) = 0$ for given $\epsilon > 0$ we have

$$|f_n(x) - f(x)| < \epsilon \text{ if } \frac{nx}{1+n^2x^2} < \epsilon$$

or if $n^2x^2 - \frac{1}{\varepsilon}nx + 1 > 0$

which requires

$$nx > \frac{1}{2\varepsilon} + \frac{1}{2}\sqrt{\frac{1}{\varepsilon} - 4}$$

We can find an upper bound for n in any interval not containing zero but the upper bound is infinite if the interval includes 0. Hence the sequence is non-uniformly convergent on any interval which includes zero.

E.g. The sequence of functions $\{f_n(x)\}$ for $x \in [a,b]$, $a \geq 0$ is uniformly convergent, where $f_n(x) = \frac{1}{x+n}$

Sol^n – Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{x+n} = 0$

for uniformly convergence we must have

$|f_n(x) - f(x)| < \varepsilon$ for sufficiently large value of n and for all $x \in [a,b]$

i.e $\left|\frac{1}{n+x}\right| = \frac{1}{n+x} < \varepsilon$

$\Rightarrow x+n > \frac{1}{\varepsilon}$ or $n > \frac{1}{\varepsilon} - x$

we can take integer just greater than $\frac{1}{\varepsilon} - a$. Hence the sequence $\{f_n\}$ is uniformly convergent.

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